

ON THE SECOND EXTERIOR POWER OF TANGENT BUNDLES OF FANO FOURFOLDS WITH PICARD NUMBER $\rho(X) \geq 2$

KAZUNORI YASUTAKE

ABSTRACT. In this paper, we classify Fano fourfolds whose the second exterior power of tangent bundles are numerically effective with Picard number greater than one.

INTRODUCTION

In the paper [7], F. Campana and T. Peternell classify smooth projective threefolds whose the second exterior power of tangent bundle $\Lambda^2 \mathcal{T}_X$ are numerically effective ("nef"). Manifolds satisfying this condition is rare in manifolds whose anticanonical divisor is nef. Such threefolds are as follows.

THEOREM 0.1 ([7], Theorem). *Let X be a projective 3-fold with $\Lambda^2 \mathcal{T}_X$ nef. Then either \mathcal{T}_X is nef or X is one of the following.*

- (1) *X is a blowing-up of \mathbb{P}^3 at a point.*
- (2) *X is a Fano 3-fold of index 2 and $b_2(X) = 1$ except for those of degree 1, which are exactly those arising as certain double covers of the Veronese cone in \mathbb{P}^6 .*

As a first step of classifying 4-folds with this condition, we treat the case where X is Fano 4-folds with Picard number $\rho(X)$ at least 2 in this paper. We get the following theorem.

MAIN THEOREM 0.2. *Let X be a Fano 4-fold with $\rho(X) \geq 2$. Assume that $\Lambda^2 \mathcal{T}_X$ is nef. Then X is the blowing-up of \mathbb{P}^4 at a point unless \mathcal{T}_X is nef.*

There is a problem about criterion of nefness of $\Lambda^q \mathcal{T}_X$ by Campana-Peternell.

PROBLEM 0.3 ([7], Problem 6.4). *Let X be a Fano manifold. Assume that $\Lambda^q \mathcal{T}_X$ is nef on every extremal rational curve. Is then $\Lambda^q \mathcal{T}_X$ already nef?*

In the proof of main theorem we can see the following theorem also holds.

THEOREM 0.4. *Let X be a Fano 4-fold with $\rho(X) \geq 2$. Assume that $\Lambda^2 \mathcal{T}_X$ is nef on every extremal rational curves in X . Then $\Lambda^2 \mathcal{T}_X$ is nef.*

2000 *Mathematics Subject Classification.* Primary 14J40; Secondary 14J10, 14J45, 14J60.

Key words and phrases. Projective manifold, second exterior power of tangent bundle, extremal contraction, Fano bundle.

ACKNOWLEDGEMENTS

The author would like to express his gratitude to Professor Eiichi Sato for many useful discussions and much warm encouragement.

NOTATION

Throughout this paper we work over the complex number field \mathbb{C} . We freely use the customary terminology in algebraic geometry. For the simplicity $\mathcal{O}(a)$ means the line bundle $\mathcal{O}(1)^{\otimes a}$. We denote the Picard number of a variety X by $\rho(X)$. We say that an extremal contraction is of (m, n) -type if the dimension of the exceptional locus is equal to m and the dimension of the image of the exceptional locus is equal to n .

1. GENERAL RESULT

Let X be a smooth n -dimensional projective variety with $\Lambda^r \mathcal{T}_X$ nef for some integer r , $1 \leq r \leq n$. Since $\det(\Lambda^r \mathcal{T}_X) = -\binom{n-1}{r-1} K_X$, the nefness of $-K_X$ implies that Kodaira dimension of X is non-positive $\kappa(X) \leq 0$. In this section we classify the case where $\kappa(X) = 0$. Main theorem of this section is the following.

THEOREM 1.1. *Let X be a smooth projective n -dimensional variety with nef vector bundle $\Lambda^r \mathcal{T}_X$ ($1 \leq r \leq n-1$) and $\kappa(X) = 0$. Then after taking some étale covering $f: \tilde{X} \rightarrow X$, \tilde{X} is isomorphic to an abelian variety.*

PROOF. If $r = 1$, this is proved in [6], Theorem 2.3. Therefore we assume that $r \geq 1$. By the nefness of $-K_X$ and $\kappa(X) = 0$, we know that the canonical line bundle K_X is a torsion line bundle. Therefore by taking some étale covering of X , we may assume that K_X is trivial. From the existence of Kähler-Einstein metric on X , we know that \mathcal{T}_X is H-semistable with respect to any ample divisor H . On the other hand, $\Lambda^r \mathcal{T}_X$ is nef vector bundle with trivial determinant. Hence $\Lambda^r \mathcal{T}_X$ is a numerically flat vector bundle i.e. $\Lambda^r \mathcal{T}_X^\vee$ is also nef. Therefore we can show that every Chern class of $\Lambda^r \mathcal{T}_X$ are numerical trivial. In particular we have

$$c_1(\Lambda^r \mathcal{T}_X).H^{n-1} = -\binom{n-1}{r-1} K_X.H^{n-1} = 0$$

and

$$c_2(\Lambda^r \mathcal{T}_X).H^{n-2} = \left\{ \binom{d}{2} c_1^2(X) + \binom{n-2}{r-1} c_2(X) \right\}.H^{n-2} = 0,$$

where $d = \binom{n-1}{r-1}$. By the triviality of K_X , we have the equality $c_2(X).H^{n-2} = 0$. Since \mathcal{T}_X is H-semistable, $c_1(X) = 0$ and $c_2(X).H^{n-2} = 0$, \mathcal{T}_X is numerically flat. Therefore from theorem 2.3 in [6], we have an étale covering $f: \tilde{X} \rightarrow X$ such that \tilde{X} is isomorphic to an abelian variety. \square

By the theorem above, we have only to consider the case where $\kappa(X) = -\infty$. This case is more difficult than the former one. In the rest of this paper we treat the case where X is a Fano 4-fold with $\rho(X) \geq 2$.

2. PROOF OF MAIN THEOREM

At first we consider the case where X is obtained by a blowing-up of smooth variety along smooth subvariety.

LEMMA 2.1. *Let X be an n -dimensional smooth Fano manifold. We assume that X is obtained by blowing-up of a smooth manifold Y along a smooth subvariety Z . If $\Lambda^2 \mathcal{T}_X$ is nef, then Y is the projective space \mathbb{P}^n and Z is a point.*

PROOF. At first we consider the case where $\dim Z \geq 1$. Let $E \cong \mathbb{P}_Z(N_{Z/Y}^\vee)$ be the exceptional divisor of the blowing-up $\phi : X = Bl_Z(Y) \rightarrow Y$. Let F be a general fiber of $\phi|_E$. Then we have an exact sequence

$$0 \rightarrow N_{F/E} \rightarrow N_{F/X} \rightarrow N_{E/X}|_F \rightarrow 0.$$

Hence we get

$$\Lambda^2 N_{F/X} \rightarrow N_{E/X}|_F \otimes N_{F/E} \rightarrow 0.$$

Combining this with the homomorphism $\Lambda^2 \mathcal{T}_X|_F \rightarrow \Lambda^2 N_{F/X} \rightarrow 0$, we obtain the surjective homomorphism of vector bundles

$$\Lambda^2 \mathcal{T}_X|_F \rightarrow N_{E/X}|_F \otimes N_{F/E} \rightarrow 0.$$

We put $c = n - 1 - \dim Z$. Since $F \cong \mathbb{P}^c$ and $N_{E/X}|_F \otimes N_{F/E} \cong \mathcal{O}_F^{\oplus c}(-1)$ is not nef, we know that $\Lambda^2 \mathcal{T}_X$ is not nef.

Next we consider the case where $\dim Z = 0$. It follows from the argument in [5] Theorem 1.1. For the convenience we write the argument. In this case there exists an extremal rational curve C such that $E.C > 0$. In fact there is a curve C' such that $C'.E > 0$ because E is effective. Since X is Fano, C' is a linear combination of extremal rational curves with non-negative real coefficient. Hence one of extremal rational curves C such that $C.E > 0$. We consider the contraction map $\varphi : X \rightarrow W$ associated with C . Then nontrivial fibers are one dimensional. Therefore from Theorem [1] and first part of this proof, φ is conic bundle which turn to be \mathbb{P}^1 -bundle by Lemma 2.9. Since $\varphi|_E : E \cong \mathbb{P}^{n-1} \rightarrow W$ is finite surjective map, Y is isomorphic to \mathbb{P}^{n-1} by Theorem 2.2. Therefore E is a section of φ and X is a projectivization of rank 2 vector bundle. We set $X \cong \mathbb{P}_W(\mathcal{E})$. Take a line $l \subseteq W$. Restrict the \mathbb{P}^1 -bundle to l we have a Hirzebruch surface $\mathbb{P}_l(\mathcal{E}|_l)$ is isomorphic to $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Hence $X \cong \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ which is a blowing-up of \mathbb{P}^n at a point. Finally we show that $\Lambda^2 \mathcal{T}_X$ is nef. Let $\pi : \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)) \rightarrow \mathbb{P}^{n-1}$ be the natural projection. Then we have the following exact sequence

$$0 \rightarrow \mathcal{T}_\pi \rightarrow \mathcal{T}_X \rightarrow \pi^* \mathcal{T}_{\mathbb{P}^{n-1}} \rightarrow 0.$$

Therefore we get the exact sequence

$$0 \rightarrow \mathcal{T}_\pi \otimes \pi^* \mathcal{T}_{\mathbb{P}^{n-1}} \rightarrow \Lambda^2 \mathcal{T}_X \rightarrow \pi^* \Lambda^2 \mathcal{T}_{\mathbb{P}^{n-1}} \rightarrow 0.$$

Let ξ be the tautological line bundle on $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1))$. Since $\mathcal{T}_\pi \otimes \pi^* \mathcal{T}_{\mathbb{P}^{n-1}} \cong 2\xi \otimes \pi^* \mathcal{T}_{\mathbb{P}^{n-1}}(-1)$ is nef, $\Lambda^2 \mathcal{T}_X$ is also nef. \square

THEOREM 2.2 ([15], Theorem 4.1). *Let X be a smooth projective variety of dimension ≥ 1 , and let*

$$f : \mathbb{P}^n \longrightarrow X$$

be a surjective map. Then $X \cong \mathbb{P}^n$.

THEOREM 2.3 ([1], Theorem 2.3). *Let $f : X \rightarrow Y$ be an elementary contraction on a smooth variety. Assume that the dimension of a fiber of f is at most one. If $\dim f(E) = \dim E - 1 = n - 2$ and f_E is equi-dimensional then both Y and $f(E)$ are non-singular, and moreover $f : X \rightarrow Y$ is the blowing-up along the smooth center $f(E)$.*

THEOREM 2.4 ([1], Theorem 3.1). *Let $f : X \rightarrow Y$ be an elementary contraction on a smooth variety. Assume that the dimension of a fiber of f is at most one. If $\dim Y = n - 1$ and f is equi-dimensional then Y is non-singular and f induces a conic bundle structure on X .*

We begin the proof of main theorem.

2.1. (2,0)-type. In this subsection we consider the case where X has a (2,0)-type extremal contraction.

THEOREM 2.5. *Let X be a Fano fourfold with (2,0)-type contraction. Then $\Lambda^2 \mathcal{T}_X$ is not nef.*

To show this, we use the following theorem due to Kawamata.

THEOREM 2.6 ([14], Theorem 1.1). *Let X be a non-singular projective variety of dimension four defined over \mathbb{C} , and let $f : X \rightarrow Y$ be a small elementary contraction. Then the exceptional locus of f is a disjoint union of its irreducible components E_i ($i = 1, \dots, n$) such that $E_i \cong \mathbb{P}^2$ and $N_{E_i/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus^2$, where N denotes the normal bundle.*

Proof of Theorem 2.5. Let $f : X \rightarrow Y$ be the (2,0)-type contraction and E an irreducible component of the exceptional locus of f . Then by Theorem 2.6 we have $E \cong \mathbb{P}^2$ and $N_{E/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus^2$. From the exact sequence

$$0 \rightarrow \mathcal{T}_E \rightarrow \mathcal{T}_X|_E \rightarrow N_{E/X} \rightarrow 0,$$

we have a surjective homomorphism of vector bundles

$$\Lambda^2 \mathcal{T}_X|_E \rightarrow \Lambda^2 N_{E/X} \rightarrow 0.$$

Since $\Lambda^2 N_{E/X} \cong \mathcal{O}_{\mathbb{P}^2}(-2)$ is not nef, $\Lambda^2 \mathcal{T}_X$ is not nef. \square

2.2. (3,2)-type. In this subsection we consider the case where X has a $(3,2)$ -type extremal contraction.

THEOREM 2.7. *Let X be a Fano fourfold with $(3,2)$ -type contraction. Then $\Lambda^2 \mathcal{T}_X$ is not nef.*

To show this, we use the following theorem due to Ando.

THEOREM 2.8 ([1], Lemma1.5 and Lemma2.2). *Let $\varphi : X \rightarrow Z$ be a Fano-Mori contraction of a smooth variety. Suppose that a fiber F of φ contains an irreducible component of dimension 1; then F is of pure dimension 1 and all component of F are smooth rational curves. If φ is birational then F is irreducible and it is a line with respect to $-K_X$.*

PROOF. By Theorem 2.8 we have a curve C such that $-K_X.C = 1$. If $\Lambda^2 \mathcal{T}_X$ is nef, this contradicts to Lemma 2.9. \square

LEMMA 2.9. *Let X be a smooth n -dimensional projective manifold with nef vector bundle $\Lambda^2 \mathcal{T}_X$. For any rational curve C in X , we have $-K_X.C \geq 2$.*

PROOF. Let $\nu : \mathbb{P}^1 \rightarrow C$ be the normalization. We set $\nu^* \mathcal{T}_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$ where $a_1 \geq a_2 \geq \cdots \geq a_n$. We have $a_1 \geq 2$ and $-K_X.C = a_1 + a_2 + \cdots + a_n$. Since $\Lambda^2 \mathcal{T}_X$ is nef, we obtain $a_{n-1} + a_n \geq 0$. In particular we have $a_{n-1} \geq 0$. Hence we get $-K_X.C = a_1 + (a_2 + a_3 + \cdots + a_{n-2}) + (a_{n-1} + a_n) \geq 2$. \square

2.3. (3,1)-type. In this subsection we consider the case where X has a $(3,1)$ -type extremal contraction.

THEOREM 2.10. *Let X be a Fano fourfold with $(3,1)$ -type contraction. Then $\Lambda^2 \mathcal{T}_X$ is not nef.*

To show this, we use the following theorem due to Takagi.

THEOREM 2.11 ([18], Main Theorem and Theorem 1.1). *Let X be a smooth 4-fold and let $f : X \rightarrow Y$ be a contraction of an extremal ray of $(3,1)$ -type. Let E be the exceptional divisor of f and $C := f(E)$. Then*

- (1) C is a smooth curve ;
- (2) $f|_E : E \rightarrow C$ is a \mathbb{P}^2 -bundle or a quadric bundle over C without multiple fibers ;
- (3) f is the blowing-up of Y along C ;
- (4) Let F be a general fiber of $f|_E : E \rightarrow C$. Then $(F, -K_X|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)), (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$ or $(\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{F}_{2,0}})$.

Proof of Theorem 2.10. Let $f : X \rightarrow Y$ be the $(3,1)$ -type contraction, E the exceptional locus of f and F a general fiber of $f|_E$. Then by Theorem 2.11 we have $(F, -K_X|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)), (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$ or $(\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{F}_{2,0}})$. Except for the

second case, we have a rational curve C in F such that $-K_X.C = 1$. In these cases $\Lambda^2\mathcal{T}_X$ is not nef by Lemma 2.9. From the exact sequence

$$0 \rightarrow \mathcal{T}_E \rightarrow \mathcal{T}_X|_E \rightarrow N_{E/X} \rightarrow 0,$$

we have the surjective homomorphism of vector bundles $\Lambda^2\mathcal{T}_X|_E \rightarrow \Lambda^2N_{E/X} \rightarrow 0$. Since $\Lambda^2N_{E/X} \cong \mathcal{O}_{\mathbb{P}^2}(-2)$ is not nef, $\Lambda^2\mathcal{T}_X$ is not nef. If $(F, -K_X|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ then X is the blowing-up of smooth manifold Y along smooth curve C . Hence $\Lambda^2\mathcal{T}_X$ is not nef by Lemma 2.1. \square

2.4. (3,0)-type. In this subsection we consider the case where X has a (3,0)-type extremal contraction.

THEOREM 2.12. *Let X be a Fano fourfold with (3,0)-type contraction. If $\Lambda^2\mathcal{T}_X$ is nef, then X is a blowing-up of the projective space \mathbb{P}^n at a point.*

To show this, we use following theorems.

THEOREM 2.13 ([1], Theorem 2.1). *Let X be a smooth projective manifold of dimension n . Let $f : X \rightarrow Y$ be an extremal contraction and E the exceptional locus of f . Assume that $\dim E = n - 1$. Let F be a general fiber of $f_E : E \rightarrow f(E)$. Then there exists a Cartier divisor L on X such that*

- (1) $\text{Im}(\text{Pic}X \rightarrow \text{Pic}F) = \mathbb{Z}[L|_F]$ and $L|_F$ is ample on F ;
- (2) $\mathcal{O}_F(-K_X) \cong \mathcal{O}_F(pL)$ and $\mathcal{O}_F(-E) \cong \mathcal{O}_F(qL)$ for some $p, q \in \mathbb{N}$.

THEOREM 2.14 ([3], Proposition 2.4 and [9]). *Let X be a smooth 4-fold whose canonical bundle is not nef. Let $\phi : X \rightarrow Y$ be a contraction of an extremal ray. If ϕ contracts an effective divisor E to a point, then E is isomorphic to either \mathbb{P}^3 , a (possibly singular) hyperquadric or a (possibly non-normal) Del Pezzo variety.*

THEOREM 2.15 ([13], Theorem 5.14 and Remark 5.15). *Let X be a projective variety with at most locally complete intersection singularities. Then there is a rational curve C on X such that $-K_X.C \leq \dim X + 1$.*

Proof of Theorem 2.12. Let E be the exceptional divisor on X . Since E is a Cartier divisor on smooth manifold X , E is a locally complete intersection variety. E is Fano by Theorem 2.14. By Theorem 2.15 we have a rational curve C in E such that $0 < -K_E.C \leq 4$. Since

$$-K_E.C = -K_X.C - E.C \geq 2 + 1 \geq 3,$$

we have $-K_E.C = 3$ or 4 . If $-K_E.C = 3$, then we get $-K_X.C = 2$ and $-E.C = 1$. By Lemma 2.16, this case does not occur. Hence we have $-K_E.C = 4$. In this case if $-K_X.C = 2$, then it is contradiction by Lemma 2.16. Therefore $-K_X.C = 3$, $-E.C = 1$. In this case by Theorem 2.3 and Theorem 2.14 we have $E \cong \mathbb{P}^3$ and $N_{E/X} \cong \mathcal{O}_{\mathbb{P}^3}(-1)$.

Therefore X is a blowing-up of smooth manifold Y at a point. By Lemma 2.1 we have $Y \cong \mathbb{P}^4$ and complete the proof. \square

LEMMA 2.16. *Let X be a smooth variety with nef vector bundle $\Lambda^2 \mathcal{T}_X$. Let C be an extremal rational curve in X such that $-K_X.C = 2$. Then the contraction map $\varphi : X \rightarrow Y$ is \mathbb{P}^1 -bundle over a smooth variety Y and C is in a fiber of φ .*

PROOF. Let $\nu : \mathbb{P}^1 \rightarrow C \subset X$ be the normalization. Let H be an irreducible component of the Hom scheme $\text{Hom}(\mathbb{P}^1, X)$ containing the morphism ν . In this case $h^* \mathcal{T}_X \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^3$ for all $h \in H$ by Lemma 2.17. Then $\dim H = n + 2$ and H is smooth. Let G be $\text{Aut}(\mathbb{P}^1)$. Since the natural action of G on $\text{Hom}(\mathbb{P}^1, X)$ induces the free action σ of G on the connected component containing H and, consequently, on H :

$$\sigma : G \times H \longrightarrow H, \sigma(g, h)x = h(g^{-1}x), g \in G, h \in H, x \in \mathbb{P}^1,$$

G also acts on $H \times \mathbb{P}^1$ as follows:

$$\tau : G \times H \times \mathbb{P}^1 \longrightarrow H \times \mathbb{P}^1, \tau(g, h, x) = (\sigma(g, h), gx), g \in G, h \in H, x \in \mathbb{P}^1.$$

Let $\text{Chow}^d X$ be the Chow variety parametrizing 1-dimensional effective cycles C of X with $(C, -K_X) = d$. Then we have a morphism $\alpha : H \rightarrow \text{Chow}^2 X$. Let Y be the normalization $Y \rightarrow \overline{\alpha(H)}$ of the closure $\overline{\alpha(H)}$ of $\alpha(H) (\subset \text{Chow}^2 X)$ in the field $k(H)^G$ of the G -invariant rational function on H . Then (Y, Γ) is the geometric quotient $\Gamma : H \rightarrow Y$ of H by G in the sense of Mumford and Y is smooth projective. We have also a G -invariant morphism:

$$F : H \times \mathbb{P}^1 \longrightarrow Y \times X, F(h, x) = (\Gamma(h), h(x)), h \in H, x \in \mathbb{P}^1.$$

Let $Z = \text{Spec}_{Y \times X}[(F_* \mathcal{O}_{H \times \mathbb{P}^1})^G]$. Then Z is the geometric quotient $H \times \mathbb{P}^1 / G$ and is a \mathbb{P}^1 -bundle $q : Z \rightarrow Y$ in the étale topology, $\dim Z = n$. Moreover the natural projection $p : Z \rightarrow X$ is a smooth morphism from n -dimensional smooth variety to n -dimensional Fano variety. Therefore p is isomorphism. \square

LEMMA 2.17. *Let X be a Fano fourfold with nef vector bundle $\Lambda^2 \mathcal{T}_X$. Let C be a rational curve in X and $\nu : \mathbb{P}^1 \rightarrow C$ the normalization. If $-K_X.C = 2$, then $\nu^* \mathcal{T}_X \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^3$.*

PROOF. By easy computation. \square

2.5. (4,1)-type. In this subsection we consider the case where X has only (4,1)-type contraction.

THEOREM 2.18. *Let X be a Fano fourfold with only fiber type contraction and having a (4,1)-type contraction. If $\Lambda^2 \mathcal{T}_X$ is nef, then X is isomorphic to the product of the projective line \mathbb{P}^1 and smooth projective threefold Y with nef tangent bundle \mathcal{T}_Y .*

To show this, we use the following theorem.

PROPOSITION 2.19 ([23], Proposition 2.2). *Let $f : X \rightarrow Y$ be a smooth morphism from a Fano manifold X . If every fiber of f is \mathbb{P}^d , then there exists a rank $(d+1)$ vector bundle \mathcal{E} on Y such that $X = \mathbb{P}_Y(\mathcal{E})$.*

Proof of Theorem 2.18. Let $f : X \rightarrow C$ be a $(4, 1)$ -type contraction. Then C is isomorphic to \mathbb{P}^1 . Let $\rho : X \rightarrow Y$ be another contraction map of extremal ray. Then the dimension of a non-trivial fiber of ρ is one. By Theorem 2.3, we know that ρ is conic bundle. By Lemma 2.9 and Theorem 2.19, we have a rank 2 vector bundle \mathcal{E} on Y such that ρ is the natural projection $\rho : X \cong \mathbb{P}_Y(\mathcal{E}) \rightarrow Y$. Take a rational curve C' in Y and put $\nu : \mathbb{P}^1 \rightarrow C'$ its normalization. In this case we have a morphism of fiber type of Hirzebruch surface $f|_S : S \cong \mathbb{P}_{\mathbb{P}^1}(\nu^*\mathcal{E}) \rightarrow C$. Therefore $\mathbb{P}_{\mathbb{P}^1}(\nu^*\mathcal{E}) \cong \mathbb{P}^1 \times \mathbb{P}^1$. By theorem 2.20, we have that \mathcal{E} is trivial up to twist. Therefore we get $X \cong Y \times \mathbb{P}^1$. We put projections $p : X \rightarrow Y$ and $q : X \rightarrow \mathbb{P}^1$. By the nefness of

$$\Lambda^2 \mathcal{T}_X \cong \Lambda^2(p^* \mathcal{T}_Y) \bigoplus (p^* \mathcal{T}_Y \otimes q^* \mathcal{T}_{\mathbb{P}^1}),$$

we have the tangent bundle \mathcal{T}_Y is nef. \square

THEOREM 2.20 ([4], Theorem 1.1). *Let \mathcal{E} be a vector bundle of rank r over a rationally connected smooth projective variety X over \mathbb{C} such that for every morphism*

$$\gamma : \mathbb{P}^1 \longrightarrow X,$$

the pullback $\gamma^ \mathcal{E}$ is isomorphic to $\mathcal{L}(\gamma)^{\oplus r}$ for some line bundle $\mathcal{L}(\gamma)^{\oplus r}$ on \mathbb{P}^1 . Then there is a line bundle ζ over X such that $\mathcal{E} = \zeta^{\oplus r}$.*

2.6. (4,2)-type. In this subsection we consider the case where X has only $(4, 2)$ -type and $(4, 3)$ -type contraction. If X has a $(4, 3)$ -type contraction, we can show that it is \mathbb{P}^1 -bundle. To show this, we use the following theorem.

THEOREM 2.21 ([12], Theorem 0.6). *Let X be a 4-dimensional smooth projective variety, R an extremal ray of X , and $g : X \rightarrow Y$ the contraction morphism associated to R . Assume that $\dim Y = 3$. Let E be a 2-dimensional fiber of g . we call*

$$l_E(R) := \min\{(-K_X.C) | C \text{ is an irreducible rational curve contained in } E\}$$

the length of R at E . It is seen that $l_E(R) = 1$ or 2 . Assume $l_E(R) = 2$. Then E is irreducible and is isomorphic to \mathbb{P}^2 , $N_{E/X} \cong \Omega_{\mathbb{P}^2}^1(1)$ and Y is smooth at P .

LEMMA 2.22. *Let X be a Fano fourfold with $(4, 3)$ -type contraction $\varphi : X \rightarrow Y$. We assume that $\Lambda^2 \mathcal{T}_X$ is nef. Then φ is \mathbb{P}^1 -bundle.*

PROOF. If φ has a two dimensional fiber F , then by Theorem 2.21 we have $F \cong \mathbb{P}^2$ and $N_{F/X} \cong \Omega_{\mathbb{P}^2}^1(1)$. We get an exact sequence of vector bundles

$$\mathcal{T}_X|_F \rightarrow N_{F/X} \cong \Omega_{\mathbb{P}^2}^1(1) \rightarrow 0.$$

Therefore we have the surjection

$$\Lambda^2 \mathcal{T}_X|_F \rightarrow \Lambda^2 \Omega_{\mathbb{P}^2}^1(1) \cong \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow 0.$$

This contradicts to the nefness of $\Lambda^2 \mathcal{T}_X$. Hence φ is equidimensional. From Theorem 2.3 and Lemma 2.9, we know that φ is \mathbb{P}^1 -bundle. \square

THEOREM 2.23. *Let X be a Fano fourfold with only (4, 2)-type and (4, 3)-type contraction. We assume that X has a (4, 2)-type contraction and $\Lambda^2 \mathcal{T}_X$ is nef. Then \mathcal{T}_X is nef.*

PROOF. Let $f : X \rightarrow S$ be a (4, 2)-type contraction. Then f is equi-dimensional and S is smooth. By lemma 2.9 and 2.16 a general fiber of f is isomorphic to \mathbb{P}^2 . By theorem 2.24 and theorem 2.19 there is a rank 3 vector bundle \mathcal{E} on S such that f is the projection $\pi : X \cong \mathbb{P}_S(\mathcal{E}) \rightarrow S$. From theorem 1.6 in [21], S is a del Pezzo surface. Since S cannot have a birational type contraction, S is isomorphic to \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$. If X has another (4, 2)-type contraction then X is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$ by Theorem 2.26. If X has two (4, 3)-type contractions, X is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ by Theorem 2.26. If X has only one (4, 3)-type contraction, then we have $\rho(X) = 2$ by Theorem 2.25. Hence S is \mathbb{P}^2 . By the list of rank 3 Fano bundle on \mathbb{P}^2 , Theorem 4.2, such a case does not exist. Therefore we have a proof. \square

THEOREM 2.24 ([11], Theorem 1.3). *Let X be a quasi-projective manifolds that admits an elementary contraction of fibre type $\varphi : X \rightarrow Y$ onto a normal variety Y such that the general fibre has dimension d . Suppose that the contraction has length $l(R) = d + 1$. If φ is equidimensional, it is a projective bundle.*

THEOREM 2.25 ([24], Theorem 2.2). *Let a manifold X of dimension n admit k different contractions of different extremal rays. If by m_i , $i = 1, 2, \dots, k$, we denote dimensions of images of these contractions, then*

$$\sum_{i=1}^k (n - m_i) \leq n.$$

THEOREM 2.26 ([17], Theorem 1.1). *A smooth complex projective variety X of dimension n is isomorphic to a product of projective spaces $\mathbb{P}^{n(1)} \times \dots \times \mathbb{P}^{n(k)}$ if and only if there exist k unsplit covering families of rational curves V^1, \dots, V^k of degree $n(1) + 1, \dots, n(k) + 1$ with $\sum n(i) = n$ such that the numerical classes of V^1, \dots, V^k are linearly independent in $N_1(X)$.*

2.7. (4,3)-type. In this subsection we consider the case where X has only (4, 3)-type contraction.

THEOREM 2.27. *Let X be a Fano fourfold with only (4, 3)-type contraction. If $\Lambda^2 \mathcal{T}_X$ is nef, then \mathcal{T}_X is nef.*

PROOF. If X has four $(4, 3)$ -type contractions then X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by Theorem 2.26 and lemma 2.9.

If X has three $(4, 3)$ -type contractions then $\rho(X) = 3$ and there is a rank 2 vector bundle \mathcal{E} on a smooth Fano threefold Z with $\rho(Z) = 2$ such that X is isomorphic to $\pi : \mathbb{P}_Z(\mathcal{E}) \rightarrow Z$. By the surjection of vector bundles

$$\Lambda^2 \mathcal{T}_X \rightarrow \pi^*(\Lambda^2 \mathcal{T}_Z) \rightarrow 0,$$

we know that $\Lambda^2 \mathcal{T}_Z$ is nef. Since X does not have a birational type contraction, we have Z is either $\mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{P}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2})$ by theorem 0.1 and theorem 3.2. If Z is $\mathbb{P}^1 \times \mathbb{P}^2$ (resp., $\mathbb{P}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2})$), we have $\mathcal{E}|_C \cong \mathcal{O}_{\mathbb{P}^1}^2(a)$ for every fiber $C \cong \mathbb{P}^1$ of natural projection $\mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ (resp., $\mathbb{P}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2}) \rightarrow \mathbb{P}^2$) since

$$2 = -K_Z.C = -K_X.\tilde{C} + (a_2 - a_1) \geq 2 + (a_2 - a_1)$$

where \tilde{C} is an extremal rational curve on X associated with C and $\mathcal{E}|_C \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ ($a_1 \leq a_2$). Therefore there is a rank 2 Fano bundle \mathcal{E}' on \mathbb{P}^2 such that \mathcal{E} is the pullback of \mathcal{E}' by the natural projection $\pi : Z \rightarrow \mathbb{P}^2$ up to twist by a line bundle. X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2})$ by using the list of Fano bundles, Theorem 4.1.

If X has two $(4, 3)$ -type contraction then $\rho(X) = 2$ and there is a rank 2 vector bundle \mathcal{E} on a smooth Fano threefold Z with $\rho(Z) = 1$ such that X is isomorphic to $\pi : \mathbb{P}_Z(\mathcal{E}) \rightarrow Z$. By the surjection of vector bundle $\Lambda^2 \mathcal{T}_X \rightarrow \pi^*(\Lambda^2 \mathcal{T}_Z) \rightarrow 0$, we know that $\Lambda^2 \mathcal{T}_Z$ is nef. Since $\rho(Z) = 1$, we have Z is either \mathbb{P}^3 , Q_3 or del Pezzo threefold of degree ≥ 2 with $\rho(Z) = 1$ from Theorem 0.1 and theorem 3.2. If Z is a del Pezzo threefold of degree ≥ 2 with $\rho(Z) = 1$, then by theorem 2.28 we have $X \cong Z \times \mathbb{P}^1$. We put the first projection $p : X \rightarrow Z$ and the second projection $q : X \rightarrow \mathbb{P}^1$. Then we have

$$\Lambda^2 \mathcal{T}_X \cong p^*(\Lambda^2 \mathcal{T}_Z) \oplus (p^* \mathcal{T}_Z \otimes q^* \mathcal{T}_{\mathbb{P}^1}).$$

From this we have the nefness of \mathcal{T}_Z and this is a contradiction. Therefore Z is either \mathbb{P}^3 or Q_3 . In this case X is isomorphic to the projectivization of null correlation bundle $\mathbb{P}(\mathcal{N})$ by the list of Fano bundles stated in Theorem 4.3 and Theorem 4.4. \square

THEOREM 2.28 ([2], Proposition 1.2). *Let X be a Fano manifold of $\rho(X) = 1$, E a rank r vector bundle on X and $V \subset \text{Hom}(\mathbb{P}^1, X)$ an unsplit family of rational curves whose locus is the whole X . Suppose that there exists an integer a such that for any $f \in V$ we have $f^* \mathcal{E} = \mathcal{O}_X(a)^{\oplus r}$. Then there exists a line bundle \mathcal{L} over X such that $\text{deg} f^* \mathcal{L} = a$ and $\mathcal{E} \cong \mathcal{L}^{\oplus r}$.*

3. VARIETIES WITH NEF TANGENT BUNDLE

In this section we list results about varieties whose tangent bundles are nef.

3.1. Surfaces with \mathcal{T}_X nef.

THEOREM 3.1 ([6], Theorem 3.1). *Let X be a smooth projective surface and assume \mathcal{T}_X to be nef. Then X is minimal and exactly one of the surfaces in the following list.*

- (1) $X = \mathbb{P}^2$.
- (2) $X = \mathbb{P}^1 \times \mathbb{P}^1$.
- (3) $X = \mathbb{P}(\mathcal{E})$, \mathcal{E} a semistable rank 2-vector bundle on an elliptic curve C .
- (4) X is abelian surface.
- (5) X is hyperelliptic.

3.2. 3-folds with \mathcal{T}_X nef.

THEOREM 3.2 ([6], Theorem 6.1 and Theorem 10.1). *Let X be a smooth projective three-fold and assume \mathcal{T}_X to be nef. Then some étale covering \tilde{X} of X belong to the following list.*

- (1) \mathbb{P}^3 .
- (2) 3-dimensional smooth quadric Q_3 .
- (3) $\mathbb{P}^1 \times \mathbb{P}^2$.
- (4) $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$.
- (5) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
- (6) $\tilde{X} = \mathbb{P}(\mathcal{E})$ for a flat rank 3-vector bundle on an elliptic curve C .
- (7) $X = \mathbb{P}(\mathcal{F}) \times_C \mathbb{P}(\mathcal{F}')$ for flat rank 2-vector bundles \mathcal{F} and \mathcal{F}' over an elliptic curve C .
- (8) $\tilde{X} = \mathbb{P}(\mathcal{E})$ for a flat rank 2-vector bundle \mathcal{E} on an abelian surface.
- (9) \tilde{X} is an abelian threefold.

3.3. 4-folds with \mathcal{T}_X nef.

THEOREM 3.3 ([8], Theorem 3.1, [16] Main Theorem and [10] Theorem 4.2). *Let X be a smooth 4-fold with \mathcal{T}_X nef. Then some étale covering \tilde{X} of X belong to the following list.*

- (1) \mathbb{P}^4 .
- (2) 4-dimensional smooth quadric Q_4 .
- (3) $\mathbb{P}^3 \times \mathbb{P}^1$.
- (4) $Q_3 \times \mathbb{P}^1$.
- (5) $\mathbb{P}^2 \times \mathbb{P}^2$.
- (6) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$.
- (7) $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2}) \times \mathbb{P}^1$.
- (8) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
- (9) $\mathbb{P}(\mathcal{E})$ with a null correlation bundle \mathcal{E} on \mathbb{P}^3 .
- (10) a Q_3 - or a flat \mathbb{P}^3 -bundle over an elliptic curve C .

- (11) a flat \mathbb{P}^2 -bundle over a ruled surface Y over an elliptic curve with \mathcal{T}_Y nef.
- (12) a flat \mathbb{P}^1 -bundle over a flat \mathbb{P}^2 - or $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle over an elliptic curve with \mathcal{T}_Y nef.
- (13) $\tilde{X} = \mathbb{P}(\mathcal{E})$ for a flat rank 3-vector bundle \mathcal{E} on an abelian surface.
- (14) $X = \mathbb{P}(\mathcal{F}) \times_A \mathbb{P}(\mathcal{F}')$ for flat rank 2-vector bundles \mathcal{F} and \mathcal{F}' over an abelian surface A .
- (15) $\tilde{X} = \mathbb{P}(\mathcal{E})$ for a flat rank 2-vector bundle \mathcal{E} on an abelian 3-fold.
- (16) abelian 4-fold.

4. FANO BUNDLE

In this section we collect results about Fano bundles (i.e. vector bundles whose projectivization are Fano) used in the proof of main theorem.

THEOREM 4.1 ([19], Theorem). *\mathcal{E} be a rank 2 Fano bundle on \mathbb{P}^2 . Then \mathcal{E} is isomorphic to one of the following up to twist by some line bundle:*

- (1) $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$
- (2) $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$
- (3) $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$
- (4) $\mathcal{T}_{\mathbb{P}^2}$
- (5) $0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_x \rightarrow 0$ where \mathcal{I}_x is the ideal sheaf of a point
- (6) stable bundle with $c_1 = 0$, $c_2 = 2$
- (7) stable bundle with $c_1 = 0$, $c_2 = 3$ and $\mathcal{E}(1)$ is spanned

THEOREM 4.2 ([20], Theorem). *Let \mathcal{E} be a rank 3 Fano bundle on \mathbb{P}^2 . Then \mathcal{E} is isomorphic to one of the following up to twist by some line bundle:*

- (1) $\mathcal{O}_{\mathbb{P}^2}^3$
- (2) $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^2$
- (3) $\mathcal{T}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}$
- (4) $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}^2$
- (5) $\mathcal{O}_{\mathbb{P}^2}^2(1) \oplus \mathcal{O}_{\mathbb{P}^2}$
- (6) $\mathcal{T}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$
- (7) $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E}_2$ where \mathcal{E}_2 is in $0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{E}_2(-1) \rightarrow \mathcal{I}_x \rightarrow 0$, \mathcal{I}_x is the ideal sheaf of a point
- (8) a bundle fitting into $0 \rightarrow \mathcal{O}_{\mathbb{P}^2}^2(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}^5 \rightarrow \mathcal{E} \rightarrow 0$
- (9) a bundle fitting into $0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^2}^4 \rightarrow \mathcal{E} \rightarrow 0$

THEOREM 4.3 ([21], Theorem 2.1). *Let \mathcal{E} be a rank 2 Fano bundle on \mathbb{P}^3 . Then \mathcal{E} is isomorphic to one of the following up to twist by some line bundle:*

- (1) $\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}$
- (2) $\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)$

- (3) $\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$
- (4) $\mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$
- (5) *the null-correlation bundle \mathcal{N}*

THEOREM 4.4 ([22], Theorem). *Let \mathcal{E} be a rank 2 Fano bundle on Q_3 . Then \mathcal{E} is isomorphic to one of the following up to twist by some line bundle:*

- (1) $\mathcal{O}_{Q_3} \oplus \mathcal{O}_{Q_3}(-1)$
- (2) *spinor bundle*
- (3) $\mathcal{O}_{Q_3} \oplus \mathcal{O}_{Q_3}$
- (4) $\mathcal{O}_{Q_3}(-1) \oplus \mathcal{O}_{Q_3}(1)$
- (5) *stable bundle with $c_1 = 0$, $c_2 = 2$*

REFERENCES

- [1] T. ANDO, On extremal rays of the higher dimensional varieties, Invent. Math. 81 (1985), 347–357.
- [2] M. ANDREATTA AND J. A. WISNIEWSKI, On manifolds whose tangent bundle contains an ample subbundle, Invent. Math. 146 (2001), 209–217.
- [3] M. BELTRAMETTI, On d-folds whose canonical bundle is not numerically effective, according to Mori and Kawamata, Annali. Mat. Pura e Appl 147 no.1 (1987), 151–172.
- [4] I. BISWAS AND J. P. P. DOS SANTOS, On the vector bundles over rationally connected varieties, C. R. Acad. Sci. Paris, Mathematique 347 (2009), 1173–1176.
- [5] L. BONAVERO, F. CAMPANA AND J. A. WISNIEWSKI, Variétés complexes dont l’écclatée en un point est de Fano, C. R. Acad. Sci. Paris, Ser. I 334 (2002), 463–468.
- [6] F. CAMPANA AND T. PETERNELL, Projective manifolds whose tangent bundles are numerically effective, Math. Ann. 289 (1991), 169–187.
- [7] F. CAMPANA AND T. PETERNELL, On the second exterior power of tangent bundles of threefolds, Compositio Math. 83 no.3 (1992), 329–346.
- [8] F. CAMPANA AND T. PETERNELL, 4-folds with numerically effective tangent bundles and second Betti numbers greater than one, Manuscripta Math. 79 (1993), 225–238.
- [9] T. FUJITA, On singular Del Pezzo varieties, Springer Lecture Note 1417 (1990), 117–128.
- [10] J. M. HWANG, Rigidity of rational homogeneous spaces, Proc. of ICM. Euro. Math. Soc. (2006), 613–626.
- [11] A. HÖRING AND C. NOVELLI, Mori contractions of maximal length, preprint
- [12] Y. KACHI, Extremal contractions from 4-dimensional manifolds to 3-folds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 24 No.1 (1997), 63–131.
- [13] J. KOLLÁR, Rational Curves on Algebraic Varieties, Ergeb. Math. Grenzgeb., 32, Springer, 1996.
- [14] Y. KAWAMATA, Small contractions of four dimensional algebraic manifolds, Math. Ann. 284 (1989), 595–600.
- [15] R. LAZARSFELD, Some applications of the theory of positive vector bundles, Springer Lecture Note 1092, 29–61.
- [16] N. MOK, On Fano manifolds with nef tangent bundles admitting 1-dimensional varieties of minimal rational tangents, Trans. of A.M.S. 357 no.7 (2002), 2639–2658.
- [17] G. OCCHETTA, A characterization of products of projective spaces, Canad. Math. Bull. 49 (2) (2006), 270–280.

- [18] H. TAKAGI, Classification of extremal contractions from smooth fourfolds of $(3, 1)$ -type, Proc. of Amer. Math. Soc. 127 No. 2 (1999), 315–321.
- [19] M. SZUREK AND J. A. WISNIEWSKI, Fano bundles of rank 2 on surfaces, Compositio Math. 76 (1990), 295–305.
- [20] M. SZUREK AND J. A. WISNIEWSKI, On Fano manifolds, which are \mathbb{P}^k -bundles over \mathbb{P}^2 , Nagoya Math. J. 120 (1990), 89–101.
- [21] M. SZUREK AND J. A. WISNIEWSKI, Fano bundles over \mathbb{P}^3 and Q_3 , Pacific journal of Math. 141 no.1 (1990), 197–208.
- [22] I. SOLS, M. SZUREK AND J. A. WISNIEWSKI, Rank-2 Fano bundles over a smooth quadric Q_3 , Pacific J. of Math. 148 no.1 (1991), 153–159.
- [23] K. WATANABE, Fano 5-folds with nef tangent bundles and Picard numbers greater than one, preprint.
- [24] J. A. WISNIEWSKI, On contractions of extremal rays of Fano manifolds, J. reine angew. Math. 417 (1991), 141–157.

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KYUSHU UNIVERSITY

FUKUOKA 819-0395

JAPAN

E-mail address: k-yasutake@math.kyushu-u.ac.jp